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Dynamic Nonlinear Modelization of Operational Supply Chain Systems

LAURA DI GIACOMO and GIACOMO PATRIZI
Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università di Roma "La Sapienza", Italy (e-mail: g.patrizi@caspur.it)

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Abstract. Supply Chain Management (SCM) is an important activity in all producing facilities and in many organizations to enable vendors, manufacturers and suppliers to interact gainfully and plan optimally their flow of goods and services. To realize this, a dynamic modelling approach for characterizing supply chain activities is opportune, so as to plan efficiently the set of activities over a distributed network in a formal and scientific way. The dynamical system will result so complex that it is not generally possible to specify the functional forms and the parameters of interest, relating outputs to inputs, states and stochastic terms by experiential specification methods. Thus the algorithm that will presented is Data Driven, determining simultaneously the functional forms, the parameters and the optimal control policy from the data available for the supply chain. The aim of this paper is to present this methodology, by considering dynamical aspects of the system, the presence of nonlinear relationships and unbiased estimation procedures to quantify these relations, leading to a nonlinear and stochastic dynamical system representation of the SCM problem. Moreover, the convergence of the algorithm will be proved and the satisfaction of the required statistical conditions demonstrated. Thus SCM problems may be formulated as formal scientific procedures, with well defined algorithms and a precise calculation sequence to determine the best alternative to enact. A "Certainty equivalent principle" will be indicated to ensure that the effects of the inevitable uncertainties will not lead to indeterminate results, allowing the formulation of demonstrably asymptotically optimal management plans.

Key words: supply chain management, simultaneous estimation and optimization, nonlinear dynamical systems

1. Introduction

Supply Chain Management (SCM) is the integration of key business processes from end users through the original suppliers that provide products, services and information that add value to customers and other stakeholders [44]. It is therefore concerned with the organization, the planning and the qualitative and quantitative determination of material and information flows both in and between facilities (vendors, plants, sites and distribution centres). It is a set of important activities in all producing facilities and in many organizations [43, 66].

These activities are carried out over time, but implementations may consider just the results which can be obtained at a given specific future moment in time (static model), or, after having evaluated the current position, determine also the changes that could be brought about at some future period in time (comparative statics).

However these approaches do not analyse what happens between 'now' and 'then'. The actual realization of these activities and the pursuit of the desired goals may flounder because of infeasibilities in the intermediary period. A model with dated variables seems preferable, which will allow the required trajectories of the flows to be followed period by period [53, 80, 82, 83].

Data driven modelling techniques will have to be applied to determine these dynamical systems representations. The functional forms to be used and the estimates of the parameters of interest so as to relate outputs to inputs, states and stochastic terms will be so complex that it is not generally possible to specify a priori these forms or determine the parameters in some simple way. Thus a data driven model must be considered, so as to determine simultaneously the functional forms, the parameters and the optimal control policy from the data available from the supply chain (accounting data). A model that uses past data to estimate the supply chain performance functions based on the chosen inputs, as a results of decisions, states and uncertainty must be applied. The estimated performance response functions, which account for the nonlinearities and stochasticity of the system, allow the specification of the decisions and policies that will optimize the resulting response functions.

The aim of this paper is, therefore, to describe an algorithm to model an SCM system formally by considering the dynamical aspects of the system, the presence of nonlinear relationships and unbiased estimation procedures to quantify the underlying relationships, leading to a nonlinear and stochastic dynamical system representation of the SCM problem. Moreover, the convergence of the algorithm will be proved and the satisfaction of the required statistical conditions demonstrated.

To determine a dynamical model of the material and information flows to solve the SCM problem at whatever level is required, historical flow data from the organization must be used, which are expressed as time series of flows. It is notoriously difficult to model vector time series because of their temporal interactions [32, 39] and to this end a coherent methodology for dynamical structures and variables must be applied [38, 70]. The phenomenon must be studied through a set of dated variables, expressed through short, noisy, nonlinear time series. For these series the determination of their short run behaviour, rather than the asymptotic behaviour, is important [24, 68].

There are many implementations of linear dynamic SCM models [5, 6, 17, 48, 71, 78, 79]. Linear dynamic system models have been suggested to regulate the bullwhip effect dynamically [18, 19], and so extend the traditional analysis [12, 45, 46].

However linear dynamical systems may not provide very realistic models of an SCM problem, since one of the features of these systems is that at any period of time, an action can be taken whose effects can be annulled at the very next period by a suitable reverse action [10]. Thus time can run in either direction, which is generally a very unrealistic property of business processes. Dynamic linear systems are unlikely, therefore, to be good representations of the given management process.

Some implementations have also been formulated with nonlinear dynamical systems of various kinds, both in a simulation and in an optimization context [30, 31, 49, 55, 56, 63, 64], but in none of these papers was the estimation problem for the functional forms and the parameters analysed or applied. This is an important requirement, as shall be shown.

Linear dynamical models, static and comparative static models, are special cases of nonlinear dynamical system and so are dominated by the latter, since they can be obtained as special cases of the latter. The nonlinear dynamical systems will therefore provide solutions which are as good, or better than the solutions obtained with the static or linear dynamical models.

Dynamic multi-item multi-level capacitated lot sizing problems, which have been extensively studied [11, 28, 29, 69], introduce further important modeling properties for SCM problems. Setup costs and machine setup times are important elements of lot sizing problems of all types and are usually sequence dependent. In some applications [74] sequence dependencies of tasks are not considered, while in other applications stochastic disturbances (e.g. breakdowns) are neglected [51, 52], which can be very important [65].

Stochastic considerations raise, once again, the problem of specifying and estimating the model to be applied. Intuitive validation of functional forms and the determination of the values to assign to parameters are very difficult to formulate, especially in nonlinear models, so appropriate estimation techniques must be applied, to obtain accurate models of the SCM problem envisaged.

For the reasons given above, in this paper a more general formulation of a SCM problem will be presented, which will consider the following elements:

1. A representation of the problem by postulating a dynamical system:
    - to produce more incisive policies, with explicit dated flows so as to ensure feasibility of the solution trajectory throughout the policy interval;
    - by using a dynamical system, which is a precise mathematical object [38] the representation of the given SCM problem is

inserted in a formal mathematical structure to ensure that the derivations from this structure will be valid.

2. Nonlinear relationships are postulated for the model:
   - for generality, from the data of the SCM problem posed, specification of the nonlinear functional forms must be envisaged;
   - so the functional form of the mathematical structure will depend on the data.
3. The solution to the SCM problem may be cast as an extremal solution:
   - the system to be solved will involve a minimization (maximization) problem or an optimal control problem.
4. The presence of stochastic elements are postulated;
   - this will require that the parameters of the relationships must be estimated by proper statistical techniques for accuracy and to avoid bias;
   - since the processes to be estimated may be nonlinear, dynamic and stochastic, the estimation must be carried out simultaneously with the optimization, so as to obtain an optimal solution in the estimation and in the control space.
5. A "Certainty Equivalent principle" [4, 13, 42, 72, 76], should be applied:
   - to render determinate the stochastic solutions that would be obtained;
   - to apply adaptive techniques or periodic reassessments of the plans, as environmental and exogenous conditions change [58].

It will be shown that such a dynamic nonlinear stochastic system formulation of an SCM model will avoid the various limitations indicated above and provide formal models, which are demonstrably valid, given certain mild conditions regarding the environment.

To the best of our knowledge the approach suggested here is new as solutions are determined to the SCM problems by a nonlinear dynamic stochastic system representation, in which the optimal control problem is solved simultaneously to determine the parameter values in the estimation space and the controls in the decision space and then applying a certainty equivalence principle.

Thus the major question in formulating the properties of this algorithm is to ensure that all its parts are well defined and coherent, which is the concern of the rest of this paper.

The outline of the paper is the following. In the next section, the properties of a dynamical system representation will be analysed to show how a supply chain problem may be recast as a nonlinear dynamical system. Further, the local analysis of time series that is required here is best handled by such an approach [24, 68, 70]. In Section 3 the specification of the system will be given, the algorithm to solve it and its statistical properties will be presented. Then, in Section 4 the mathematical properties of the algorithm will be formulated, as well as the convergence conditions. In Section 5 the appropriate conclusions will be drawn.

2. Dynamical Systems and their Properties

Mathematical System Theory deals essentially with the study of the dynamical relationships of systems under various conditions, more general than those which define difference and differential equation systems [81]. A Dynamical System is a precise mathematical object, [37, 38] and given the flows of the activities of the phenomenon the input–output relationships must be estimated by appropriate estimation methods.

Not every relationship can be modelled by mathematical system theory, since a representation which is nonanticipatory is required [38], while the condition that the functionals be sufficiently smooth which was previously required [38], may be wavered.

Dynamical Systems have been defined at a high level of generality, to refine concepts and perceive unity in a diversity of applications and by appropriate modelling, whole hierarchies of phenomena can be represented as systems defined at different levels. This is very important in living organisms [53], complex machines (computers, vehicles) [83] and other phenomena, [80, 82]. Again SCM exhibits hierarchies of representations and dynamical interactions.

DEFINITION 2.1. [38]. A Dynamical System is a composite mathematical object defined by the following axioms:

1. There is a given time set T, a state set X, a set of input values U, a set of acceptable input functions $\Omega = \omega : \Omega \to U$, a set of output values Y and a set of output functions $\Gamma = \gamma : \Gamma \to Y$.
2. (Direction of time). T is an ordered subset of the reals.
3. The input space Ω satisfies the following conditions.
   (a) (Nontriviality). Ω is nonempty.
   (b) (Concatenation of inputs). An input segment $\omega_{(t_1,t_2]}$, $\omega \in \Omega$ restricted to $(t_1, t_2] \cap T$. If $\omega, \omega' \in \Omega$ and $t_1 < t_2 < t_3$ there is an $\omega'' \in \Omega$ such that $\omega''_{(t_1,t_2]} = \omega_{(t_1,t_2]}$ and $\omega''_{(t_2,t_3]} = \omega'_{(t_2,t_3]}$.
4. There is a state transition function $\varphi : T \times T \times X \times \Omega \to X$ whose value is the state $x(t) = \varphi(t; \tau, x, \omega) \in X$ resulting at time $t \in T$ from the initial state $x = x(\tau) \in X$ at the initial time $\tau \in T$ under the action of the input $\omega \in \Omega$. φ has the following properties:
   (a) (Direction of time). φ is defined for all $t \geqslant \tau$, but not necessarily for all $t < \tau$.

(b) (Consistency). $\varphi(t;t,x,\omega)=x$ for all $t\in T$, all $x\in X$ and all $\omega\in\Omega$.
(c) (Composition property). For any $t_1<t_2<t_3$ there results:

$$\varphi(t_3;t_1,x,\omega)=\varphi(t_3;t_2,\varphi(t_2;t_1,x,\omega),\omega)$$

for all $x\in X$ and all $\omega\in\Omega$.
(d) (Causality). If $\omega,\omega'\in\Omega$ and $\omega_{(\tau,t]}=\omega'_{(\tau,t]}$ then $\varphi(t;\tau,x,\omega)=\varphi(t;\tau,x,\omega')$.

5. There is a given readout map $\eta:T\times X\rightarrow Y$ which defines the output $y(t)=\eta(t,x(t))$. The map $(\tau,t]\rightarrow Y$ given by $\sigma\mapsto\eta(\sigma,\varphi(\sigma,\tau,x,\omega))$, $\sigma\in(\tau,t]$, is an output segment, that is the restriction $\gamma_{(\tau,t]}$ of some $\gamma\in\Gamma$ to $(\tau,t]$.

The following mathematical structures in Definition 2.1 will be indicated by:

- the pair $(t,x), t\in T, x\in X \quad \forall t$ is called an event;
- the state transition function $\varphi(x_t,u_t)$ is called a trajectory.

Consider a department, a plant or a firm, see [21] for specific implementations. The set U, defined conveniently may differ from application to application, indicating the dated quantities of raw materials, energy, labour and so on: activities applied on a specific set of machines, department, plant or firm. The historical data is obtained from analytical and budget accounts, the 'Gozinto' charts [8], as well as many Information Technology (IT) tools. Consequent to this, through the input and output functions a set Y of dated quantities in output is obtained. As the process may be highly nonlinear with marked lags, intermediary vectors, indicated as states are used with an opportune transition function. The states are related to the outputs obtained in time. The effect of all the activities at a moment t on the state of the system is called an event. A trajectory may be understood as the graph of the state as a consequence of the variation in time.

This notion of a dynamical system is very general and a certain degree of additional structure may be imposed, so that the results gain specificity, without depriving the modellisation of much of its intrinsic interest.

As defined in the above definition, the state transition functional form and the given readout map are assumed not to change in structure over time. That is the functional form $\varphi(.)$ and $\eta(.)$ and eventually an additional performance function $c(.)$ are not time dependent, although time may be an argument of these functions. This implies that the functional forms are not evolving over time in the sense that they are consistent over time. Thus another way to model phenomena is through dynamical systems in the input/output sense.

DEFINITION 2.2. A Dynamical System in an input/output sense is a composite mathematical object defined as follows:

1. There are given sets T, U, Ω, Y and Γ satisfying all the properties required by Definition 2.1.
2. There is a set A indexing a family of functions

$$\mathcal{F} = \{f_\alpha : T \times \Omega \to Y, \alpha \in A\}$$

each member of $\mathcal{F}$ is written explicitly as $f_\alpha(t, \omega) = y(t)$ which is the output resulting at time t from the input ω under the experiment α. Each f_α is called an input/output function and has the following properties:

(a) (Direction of time). There is a map $\iota : A \to T$ such that $f_\alpha(t, \omega)$ is defined for all $t \geqslant \iota(\alpha)$.
(b) (Casuality) Let $\tau, t \in T$ and $\tau < t$ If $\omega, \omega' \in \Omega$ and $\omega_{(\tau,t]} = \omega'_{(\tau,t]}$, then $f_\alpha(t, \omega) = f_\alpha(t, \omega')$ for all α such that $\tau = \iota(\alpha)$.

While the input/output approach may determine a family of functions, the state space approach represents the trajectories in the way indicated, through a unique function, so the latter approach is intuitively more appealing, especially in applications. However, both representations show the relationships of the time series of the single inputs on the state and the outputs. The first representation defines a unique mapping, while the second representation does not.

The representations are equivalent. It is easy to transform a given system from a state space formulation to an input/output formulation and vice versa [2, 38], so each may be used as convenience suggests.

Of course, by imposing suitable smoothness conditions on the system, it can be represented as a system of differential equations and solved by standard techniques. To this end:

DEFINITION 2.3. A dynamical System is smooth if and only if:

1. $T = \mathbf{R}$ the real numbers (with the usual topology).
2. X and Ω are topological spaces.
3. The transition map φ has the property that

$$(\tau, x, \omega) \mapsto \varphi(.; \tau, x, \omega)$$

defines a continuous map $T \times X \times \Omega \to \mathbf{C}^1(T \to X)$, where $\mathbf{C}^1(T \to X)$ denotes the family of functions that are once continuously differentiable.

It is now possible to indicate when a Dynamical System defined above can be solved as a system of differential equations, by making use of well-known classical techniques [14].

THEOREM 2.1 [38]. *Let a Dynamic System, in the sense of Definition* 2.1 *which is smooth, in the sense of Definition* 2.3, *possess these further characteristics:*

1. $T = \mathbf{R}$, X and U are normed spaces,
2. Ω is the normed space of continuous functions $T \to U$ with $\|\omega\| = \sup_{t \in T} \|u(t)\|$
3. $\varphi(.; \tau, x, \omega) \in \mathbf{C}^1(T \to X)$ for each τ, x and ω and the map $T \times X \times \Omega \to X$ given by $(\tau, x, \omega) \mapsto \dot{\varphi}(t; \tau, x, \omega)$ is continuous for each t, with respect to the product topology.

Then the transition function φ of the smooth dynamic system is a solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, \pi_t \omega)$$

where the operator π_t is a map $\Omega \to U$ given by $\omega \mapsto u(t) = \omega(t)$.

In control theory there are many applications of differential systems of equations to process control [25, 26, 62, 77]. Thus this characterization provides an equivalence between the types of systems that are of concern in SCM and those representable by systems of ordinary and/or partial differential equations.

It cannot be assumed generally that a Dynamical System satisfies the conditions of smoothness, nor that it will meet the necessary and sufficient conditions for an optimal control to exist. Thus in general, the Dynamical Systems to be dealt with may have an awkward structure but through the combined estimation and optimization approach a sufficiently good approximation may be obtained with the required characteristics [61].

A sufficiently general representation of a dynamic system may be formulated by applying Definition 2.1, recalling the equivalence of an input–output system and a system in state form:

$$x_{t+1} = \varphi(x_t, u_t), \tag{1}$$

$$y_t = \eta(x_t), \tag{2}$$

where $x_t \in X \subseteq R^r$ may simply be taken as a r-dimensional vector in an Euclidean space X, indicating the state of the system at time t, $u_t \in U \subseteq R^q$ may be taken as a q-dimensional vector in an Euclidean subspace U of control variables and $y_t \in Y \subseteq R^p$ is a p-dimensional vector in an Euclidean space Y of output variables, in line with Definitions 2.1, 2.2 or 2.3.

The fundamental problem posed by such system is to determine suitable control policies through time to obtain various types of solutions, such as: optimal solutions, equilibrium solutions, periodic solutions, quasi-periodic solutions and eventual chaotic solutions (perhaps to be avoided) [57] and to be able to compare them before executing the proposed activities. Thus a plethora of conditional trajectories can be formulated and compared, so that the best, according to the firm's aims, can be considered. To this end a number of concepts are required.

The definition of a dynamical system is based on defining an intermediary set of states and a transition function or a family of functions. Neither of these constructions is unique, so if it is desired to represent a SCM system by such structures, equivalence of the possible structures must be shown.

DEFINITION 2.4. Given two states x_{t_0} and $\hat{x}_{t_0}$ belonging to systems S and $\hat{S}$ which may not be identical, but have a common input space Ω and output space Y, the two states are said to be equivalent if and only if for all input segments $\omega_{[t_0,t)} \in \Omega$ the response segment of S starting in state x_{t_0} is identical with the response segment of $\hat{S}$ starting in state $\hat{x}_{t_0}$; that is

$$\begin{aligned} x_{t_0} \cong \hat{x}_{t_0} \Leftrightarrow\, & \eta(t, \varphi(x_{t_0}, \omega_{[t_0,t)})) \\ &= \hat{\eta}(t, \hat{\varphi}(\hat{x}_{t_0}, \omega_{[t_0,t)})) \quad \forall t \in T,\ t_0 \leqslant t, \quad \forall\ \omega_{[t_0,t)} \in S, \hat{S}. \end{aligned} \tag{3}$$

The systems S and $\hat{S}$ may be two models of a SCM system solved with different control policies, or various alternative models of the phenomenon.

DEFINITION 2.5. A system is in reduced form if there are no distinct states in its state space which are equivalent to each other.

DEFINITION 2.6. Systems S and $\hat{S}$ are equivalent $S \equiv \hat{S}$ if and only if to every state in the state space of S there corresponds an equivalent state in the state space of $\hat{S}$ and vice versa.

A number of important questions must be asked of the system description of the SCM representation:

- Can a certain state $s^* \in S$ be reached from the present state, or if the dynamical system attains a given state x_0 at time 0 can it also be made to reach a certain state x^*. Evidently it is required to determine the set of states reachable from a specific state x_t.

- Can a dynamical system be driven to a given state by an input u. Thus controllability is concerned with the connectedness properties of the system representation.
- Reachability and controllability lead naturally to the determination of a dynamical system's observability, which provides the conditions to determine the given actual state uniquely.
- The stability of the system is important since it provides conditions on the way the trajectories will evolve, given a perturbation or an admissible control.

These conditions are very important, since they allow trajectories to be defined, the initial point of trajectories to be determined and their stability properties to be derived. Moreover they can be applied at any moment in time to determine if the goals of the SCM are still attainable and at what cost. Reachability, controllability and stability are seldom formally examined and yet at every period exogenous events can arise to nullify even the best plan formulated, so these are important instruments for SCM.

DEFINITION 2.7. Given a state $x^* \in M \subseteq X$, it is reachable from the event (t_0, x_0) at time T if there exists a bounded measurable input $u_t \in \Omega$ such that the trajectory of the system satisfies:

$$x_{t_0} = x_0, \tag{4}$$
$$x_T = x^* \qquad \forall x_{t_0} \in M, \quad 0 \leqslant t \leqslant T. \tag{5}$$

The sets of states reachable from x_{t_0} is denoted by:

$$\Re(x_{t_0}) = \bigcup_{0 \leqslant T \leqslant \infty} \{x_T | x_T \quad \text{reachable at time } T\} \tag{6}$$

the system is reachable at x_{t_0} if $\Re(x_{t_0}) = M$ and reachable if $\Re(x_{t_0}) = M$ $\forall x \in M$.

DEFINITION 2.8. A system is locally reachable at x_{t_0} if for every neighbourhood $N(x_{t_0}, h)$ of x_{t_0}, $\Re(x_{t_0}) \cap N_{x_0}$ is also a neighbourhood of x_{t_0} with the trajectory from the event (t_0, x_{t_0}) to $\Re(x_{t_0}) \cap N_{x_0}$ lying entirely within N_{x_0}. The system is locally reachable if it is locally reachable for each $x \in M$.

These definitions lead to an important property for many systems, namely that reachability may not be symmetric, that is: if x_T is reachable from x_{t_0} the converse may not hold. Thus a weaker notion of reachability, which is always found in linear systems, may be opportune.

DEFINITION 2.9. Two states x^* and $\hat{x}$ are weakly reachable from each other if and only if there exist states $x^0, x^1, \ldots, x^k \in M$ such that $x^0 = x^*, x^k = \hat{x}$ and either x^i is reachable from x^{i-1} or x^{i-1} is reachable from x^i ($\forall 1 = 1, 2, \ldots, k$). The system is weakly reachable if it is weakly reachable from every $x \in M$.

THEOREM 2.2. *The following implications apply:*

- *If the system is locally reachable then it is reachable;*
- *if the system is reachable then it is weakly reachable.*

Proof. Immediate from the definitions. □

COROLLARY 2.1. *For constant linear systems the following are equivalent:*

- *a system is locally reachable if and only if it is reachable;*
- *a system is reachable if and only if it is weakly reachable.*

Proof. Immediate from the definitions. □

DEFINITION 2.10. State x_{t_0} of a system is controllable if and only if there exists a $u \in \Omega$ such that:

$$\varphi(t; t_0, x_{t_0}, u) = \emptyset. \tag{7}$$

The system is said to be controllable if and only if every state of the system is controllable.

THEOREM 2.3. *A system which is controllable and in which every state is reachable from the zero state (∅) is strongly connected.*

Proof. Follows from Definitions 2.10 and 2.7 (see [38]). □

DEFINITION 2.11.

- A simple experiment is an input/output pair $(u_{[t_0,t)}, y_{[t_0,t)})$ that is, given the system in an unknown state an input $u_{[t_0,t)}$ is applied over the interval of time (t, t_0) and the output $y_{[t_0,t)}$ is observed.
- A multiple experiment of size N consists of N input/output pairs $(u^i_{[t_0,t)}, y^i_{[t_0,t)})$ $i = 1, 2, \ldots, N$ where on applying on the ith realization of the N systems the input $(u^i_{[t_0,t)})$ the ith output $y^i_{[t_0,t)}$ is observed.

DEFINITION 2.12. A system is simply (multiply) observable at state x_{t_0} if and only if a simple experiment (a multiple experiment) permits the determination of that state uniquely.

DEFINITION 2.13.

- Two systems are simply equivalent if it is impossible to distinguish them by any simple experiment.
- Two systems are multiply equivalent if it is impossible to distinguish them by any multiple experiment.

THEOREM 2.4. *If two systems are simply equivalent and strongly connected, then they are multiply equivalent.* □

THEOREM 2.5. *If two systems are multiply equivalent then they are equivalent* (*Definition* 2.6). □

DEFINITION 2.14. A system is initial-state determinable if the initial state x_0 can be determined from an experiment on the system started at x_0.

THEOREM 2.6. *A system is in reduced form if and only if it is initial-state determinable by an infinite multiple experiment.* □

Definitions 2.11–2.14 and the results 2.4–2.6 formally justify the possibility of defining one or more representations of the dynamical system considered at a chosen level of detail. Notice however the distinction between systems that are simply equivalent and multiply equivalent. This distinction is crucial, if dynamical systems are considered, while with comparative static models, the distinction does not apply. This is one of the many reasons that one should insist on solving SCM dynamic estimation problems with a data driven formulation.

It is usual, since stability for nonlinear systems is an equilibrium concept, to treat such systems by a representation as a continuous time systems, i.e. a smooth system as in Definition 2.3. It is also usual to consider the system as an autonomous system, with no input. Such a system can also be considered as an approximation to the more complex real system.

Thus consider:

$$\dot{x} = \varphi(x, t), \tag{8}$$

$$x(t_0) = x_0 \tag{9}$$

an autonomous nonlinear system, while x_0 is the initial state of the system.

DEFINITION 2.15. The function φ is said to be locally Lipschitz continuous in x if for some $h > 0$ there exists an $l \geqslant 0$ such that:

$$|\varphi(x_1, t) - \varphi(x_2, t)| \leqslant l\,|x_1 - x_2| \quad \forall x_1, x_2 \in N(0, h), t \geqslant 0. \tag{10}$$

The constant l is called the Lipschitz constant.

Without loss of generality it can be assumed that an equilibrium point for the system is the origin, i.e. $x^* = 0$. We will assume this is so in the rest of this section.

DEFINITION 2.16. x^* is said to be an equilibrium point of (8) if $\varphi(x^*, t) \equiv 0$, $\forall t \geqslant 0$.

THEOREM 2.7. *If* $x = 0$ *is an equilibrium point of system* (8)*,* φ *is Lipschitz continuous in* x *with Lipschitz constant* l *and piecewise constant with respect to* t *then the solution of* $x(t)$ *satisfies:*

$$|x_0| e^{l(t-t_0)} \geqslant |x(t)| \geqslant |x_0| e^{-l(t-t_0)} \quad \forall x(t) \in N(0, h), \quad \forall t \geqslant t_0. \tag{11}$$

Proof. see [70] □

DEFINITION 2.17. The equilibrium point $x = 0$ is called a stable equilibrium point of the system (8) if for all t_0, $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that:

$$|x_0| < \delta(t_0, \epsilon) \Rightarrow |x(t)| < \epsilon \quad \forall t \geqslant t_0. \tag{12}$$

The solution of the dynamic system given in Equations (1) and (2) may be determined in a number of different ways, depending on the structure of the functions that are given, see [9, 35, 38, 67].

In general, the dynamical system representation of a SCM system, permits to verify.

- The dynamic system representation can be identified, i.e. the functional form of the relationships is determined and the optimal value of the parameters for the chosen form are estimated.
- The optimal control which determines the final event is reachable, and the system must be controllable throughout the sequence of events comprising the trajectory.
- The system should be observable throughout in case remedial action should be taken, which would otherwise be impossible and to ensure that the estimation may be carried out.
- The given solution is stable, so that small perturbations will not give rise to explosive perturbations or to chaotic behaviour.

If these conditions are not verified, this will suggest strategic changes to the SCM system or a profound modification of policies: aspects which are difficult to determine in advance.

Computationally, these aspects are handled by adding appropriate constraints in the mathematical program which will be formulated in Section

3 [20, 22, 59]. As these aspects are rather specialized and add no additional theoretical points to the specification of the algorithm, they will not be dealt with here, although computationally they are important.

The specification of the model that has been adopted appears to allow correct application to policy determination in SCM systems, as well as to determine its control functions and many other aspects.

The advantage of such an approach is that through the conditions that can be formulated regarding the reachability, the controllability, the observability and the stability of such systems, crucial questions which are invariably posed by management can be answered.

The data driven approach which is adopted, does not impose a priori restrictions on any aspect of the representation, except for the mild limitations indicated in the definitions above. Thus the restrictions, such as linearity, or stationarity of time series, which lead to incorrect policy formulation or a limited analysis of important events, need not occur in this approach.

3. Description of the Algorithm

Consider the monitoring of a set of activities in time of a Supply Chain at a given level of aggregation, which may be at the department, plant or firm level, or a hierarchical system developed through all these organizational structures. Although the accuracy of the representation may depend on the sampling strategy and the time interval, these aspects will not be considered here.

Thus a given finite dimensional estimation and optimization problem will be considered, which may well be nonlinear and dynamic [9].

Consider the data set of a phenomenon consisting of measurements (y_t, x_t, u_t) over $(t = 1, 2, \ldots, N)$ periods, where it is assumed, that $y_t \in R^p$ is a p-dimensional vector, while $x_t \in R^r$ is a r-dimensional vector of explanatory or state variables of the dynamic process of dimension. Also, u_t is a q-dimensional vectors of control variables. It is desired to determine functional forms $\varphi: R^{r+q} \to R^r$ and $\eta: R^r \to R^p$ and a set of suitable coefficients $\Theta \in R^m$ such that:

$$\text{Min} \quad J = \sum_{t=N+1}^{\mathcal{T}} c(x_t, u_t, y_t), \tag{13}$$

$$x^{t+1} = \varphi(x_t, u_t, y_t, w_t) \quad \forall t = T+1, \ldots, \mathcal{T}-1, \tag{14}$$

$$y_t = \eta(x_t, u_t, v_t) \quad \forall t = T+1, \ldots, \mathcal{T}, \tag{15}$$

where w_t, v_t are stochastic processes also to be determined.

Equation (13) is the objective function for the supply chain and (14) and (15) in a state space formulation and a similar representation may be adopted for the the input–output formulation [38, 73].

The systems (13)–(15) could be estimated by a maximum likelihood method so as to minimize the random errors, indicated by $w_t \in R^r$ and $v_t \in R^p$ such that they will have minimum variance and zero mean value and then on the quantified model the optimal control problem could be solved, usually through an appropriate optimization problem [77].

However, for this type of model with serially correlated disturbances, which are also correlated with the control variables, its estimation will be biased and the necessary least squares properties to ensure an asymptotically correct estimate may only be fulfilled in exceptional cases. Thus the two-stage approach, indicated above, is inappropriate [34].

It is important to apply a suitable data driven statistical method to determine the most appropriate statistical form and the most precise values of the parameters, as when implemented correctly with regard to an accurately specified functional form. Such a method will provide estimates of parameters that have the following properties [1, 36, 51]:

1. The parameter estimates are unbiased, this means that:
   - as the size of the data set grows larger, the estimated parameters tend to their true values.
2. The parameter estimates are consistent, which require the following conditions to be satisfied:
   - The estimated parameters are asymptotically unbiased,
   - The variance of the parameter estimate must tend to zero as the data set tends to infinity.
3. The parameter estimates are asymptotically efficient,
   - the estimated parameter is consistent;
   - the estimated parameter has smaller asymptotic variance as compared to any other consistent estimator.
4. The residuals have minimum variance, which will require to ensure that this is so:
   - the variance of the residuals must be minimum;
   - the residuals must be homoscedastic;
   - the residuals must not be serially correlated.
5. The residuals are unbiased (have zero mean):
6. The residuals have a noninformative distribution (usually, the Gaussian distribution). If the distribution of the residuals is informative, the extra information could somehow be obtained, reducing the variance of the residuals, their bias, etc. with the result that better estimates are obtained.

In short, through correct implementation of statistical estimation techniques the estimates are as close as possible to their true values, all the information that is available is applied and the uncertainty surrounding the estimates and the data fit is reduced to the maximum extent possible. Thus the estimates of the parameters, which satisfy all these conditions, are the 'best' possible in a 'technical sense' [1].

Instead, suppose that all the statistical properties that a given estimate must fulfill are set up as constraints to the maximum likelihood problem to be solved, then the parameters are defined implicitly by this optimization problem, which can be inserted into the optimal control system for the policy determination, so that statistically correct estimates will always result. Thus the solution yielding the best policy can be chosen, where $N+1,\ldots,\mathcal{T}$ is the forecast period, by solving an optimization formulation of this complex problem [75]. By recursing on the specifications, i.e. by changing the functional form, better and better fits can be obtained. At each iteration, the best combination of parameterization and policy is obtained.

The unknowns to be determined are the input and output variables considered and the parameters of the functional form specified in the current iteration, indicated as $\Theta=\{\theta_1,\theta_2\}\subset R^m$, respectively, for (14) and (15). Note that m may be much larger than $2r+q+p+1$ the number of variables present in each system, since the system is nonlinear.

The mathematical program will be formulated with respect to the residual variables, but it is immediate that for a given functional form, the unknown parameters will be specified and thus the unknowns of the problem will also be defined and available. Thus the mathematical program is fully specified for each functional form to be considered.

Using the notation given above, the residual terms are given from the Equations (14) and (15) as:

$$w_i=\hat{x}_{i+1}-\varphi(\hat{x}_i,\hat{u}_i,\hat{y}_i:\theta_1) \quad i=1,2,\ldots,N, \tag{16}$$

$$v_i=\hat{y}_{i+1}\eta(x_i,u_i,v_i:\theta_2) \quad i=1,2,\ldots,N, \tag{17}$$

where $\hat{.}$, as usual indicates the historical values of a variable and thus suitable values of θ_1,θ_2 must be determined by the mathematical program, such that all the constraints expressed in terms of $w_i,v_i \quad \forall i$ are specified.

The homoscedasticity condition on the residuals is obtained by regressing the original variables of the problem, indicated by the data matrix Ψ, on the normalised square of the residuals, which are indicated, respectively, by: g_w,g_v. This leads to a set of nonlinear equations in the squared residuals to be determined. The χ^2 test is applied at a confidence level of $(1-\alpha)$ with $m-1$ degrees of freedom and a significance level of α [7].

The combined model to be solved with the notation given above, by a suitable optimization routine is the following:

$$\text{Min} \quad J = \sum_{i=N+1}^{\mathcal{T}} c(x_i, u_i, y_i), \tag{18}$$

$$x_{i+1} = \varphi(x_i, u_i, y_i, w_i : \theta_1), \tag{19}$$

$$y_{i+1} = \eta(x_i, u_i, v_i : \theta_2), \tag{20}$$

$$\frac{1}{N}\sum_{i=1}^{N} w_i = 0, \tag{21}$$

$$\frac{1}{N}\sum_{i=1}^{N} v_i = 0, \tag{22}$$

$$\frac{1}{N}\sum_{i=1}^{N} w_i^2 \leqslant k_w, \tag{23}$$

$$\frac{1}{N}\sum_{i=1}^{N} v_i^2 \leqslant k_v, \tag{24}$$

$$-\epsilon_0 \leqslant \frac{1}{N}\sum_{i=1}^{N} v_i w_i \leqslant \epsilon_0, \tag{25}$$

$$-\epsilon_1 \leqslant \frac{1}{N}\sum_{i=1}^{N} w_i w_{i-1} \leqslant \epsilon_1, \tag{26}$$

$$-\epsilon_2 \leqslant \frac{1}{N}\sum_{i=1}^{N} v_i v_{i-1} \leqslant \epsilon_2, \tag{27}$$

$$-\epsilon_3 \leqslant \frac{1}{N}\sum_{i=1}^{N} v_i w_{i-1} \leqslant \epsilon_3, \tag{28}$$

$$-\epsilon_4 \leqslant \frac{1}{N}\sum_{i=1}^{N} w_i v_{i-1} \leqslant \epsilon_4, \tag{29}$$

$$\ldots\ldots\ldots \quad \ldots \quad \ldots\ldots$$

$$-\epsilon_{2s} \leqslant \frac{1}{N}\sum_{i=1}^{N} v_{i-s} w_{i-s} \leqslant \epsilon_{2s}, \tag{30}$$

$$-\epsilon_{2s+1} \leqslant \frac{1}{N}\sum_{i=1}^{N} w_i w_{i-s} \leqslant \epsilon_{2s+1}, \tag{31}$$

$$-\epsilon_{2s+2} \leqslant \frac{1}{N}\sum_{i=1}^{N} v_i v_{i-s} \leqslant \epsilon_{2s+2}, \tag{32}$$

$$-\epsilon_{2s+3} \leqslant \frac{1}{N}\sum_{i=1}^{N} v_i w_{i-s} \leqslant \epsilon_{2s+3}, \tag{33}$$

$$-\epsilon_{2s+4} \leqslant \frac{1}{N}\sum_{i=1}^{N} w_i v_{i-s} \leqslant \epsilon_{2s+4}, \tag{34}$$

$$\frac{1}{2} g_w^T \Psi(\Psi^T \Psi)^{-1} \Psi^T g_w - \frac{N}{2} \leqslant \chi^2_{1-\alpha:p-1}, \tag{35}$$

$$\frac{1}{2} g_v^T \Psi(\Psi^T \Psi)^{-1} \Psi^T g_v - \frac{N}{2} \leqslant \chi^2_{1-\alpha:p-1}, \tag{36}$$

$$-\epsilon_{2r+1} \leqslant \frac{1}{N}\sum_{i=1}^{N} w_i^{2r+1} \leqslant \epsilon_{2r+1} r = 3, 4, \ldots, \tag{37}$$

$$\frac{1}{N}\sum_{i=1}^{N} w_i^{2r} \leqslant \frac{2r!}{r!2^r}\sigma_w^{2r} r = 3, 4, \ldots, \tag{38}$$

$$-\epsilon_{2r+1} \leqslant \frac{1}{N}\sum_{i=1}^{N} v_i^{2r+1} \leqslant \epsilon_{2r+1} r = 3, 4, \ldots, \tag{39}$$

$$\frac{1}{N}\sum_{i=1}^{N} v_i^{2r} \leqslant \frac{2r!}{r!2^r}\sigma_v^{2r} r = 3, 4, \ldots, \tag{40}$$

$$x_i \in X, \quad y_i \in Y, \quad u_i \in U, \quad w_i \in W, \quad v_i \in V. \tag{41}$$

We shall show that the conditions indicated above are met for an optimal solution of the programs (18)–(41).

The abstract model of the dynamical systems (13)–(15) given by the system of Equations (19) and (20), is to be optimized with regard to a given merit function (18) such that the sum of squares of the residuals to be less than a critical value k_w, k_v which can be decreased by dichotomous search at every iteration, until the problem does not yield a feasible solution.

The least values obtained for these parameters, while retaining a feasible solution to the whole problem, are equivalent to a minimization of the statistical estimation error and of a maximization of the maximum likelihood, under appropriate distributional assumptions concerning the residuals.

THEOREM 3.1. *Let the constrained minimization problems* (18)–(41) *have an optimal solution, then the residuals* $\{w_i | i \in \{1, 2, \ldots, N\}\}$ $\{v_i | i \in \{1, 2, \ldots, N\}\}$ *have zero mean, are serially uncorrelated and homoscedastic with finite minimum variance.*

Proof. The optimal solution must satisfy all the constraints of the problem, and assume a minimal value of the objective function. Satisfaction of (21) and (22) ensure that the mean of the residuals be null, (25)–(34) force the residuals to be serially uncorrelated up to lag s, while (35) and (36) ensure the homoscedasticity of the residuals. The existence of a solution ensures that a finite minimum has been identified for the value of the objective function. If this value is just a local minimum, by adding suitable upper bounding constraints, lower local minima can be determined, until the global minimum has been generated. □

The theorem states that Conditions 4 and 5 hold for the model and the data.

Moreover to ensure that these conditions hold throughout the possible variation of the independent variables, the residuals must be homoscedastic [41] and thus satisfy (35) and (36).

Further, all the serial correlations between the residual are not significantly different from zero, as enforced by the constraints (25)–(35).

The next lemma and in particular the corollaries which follow prove that conditions 2 and 1 above, given the results of Theorem 3.1, hold and so the estimates of the parameters are consistent and unbiased.

LEMMA 3.1. [16]. *Any rational function or power of a rational function of the sample moments, converges in probability to a constant, obtained by substituting throughout the corresponding population moments, provided that the latter exists and that the resulting expression is well defined.* □

COROLLARY 3.1. *Let the constrained minimization problems* (18)–(41) *have an optimal solution, with minimum values for the variances of the residuals, as the sample size increases, then the constraints* (21)–(40) *will tend to their constant population values.*

Proof. Suppose that the constrained optimization problem has optimal solutions as the sample size increases then, the sample moments will converge in probability to their population values, by Lemma 3.1.

The constraints (21)–(34) as well as (37)–(40) are sample moments, so they will converge in probability to their population values, the ones indicated by (21)–(34) to zero and of the second group, those representing the odd moments of the distribution, indicated by (37) and (39), will converge in probability to their population value of zero, while the even moments will converge in probability to their population values. □

Specifically, the definition of an unbiased estimator is:

DEFINITION 3.1. Let $\tilde{\theta}_n$ be an estimator determined for a sample of size n of the population value θ in the functional form $f(x_i, \theta)$; $\tilde{\theta}_n$ is said to be an unbiased estimator if $E\{\tilde{\theta}_n\} = \theta$.

COROLLARY 3.2. *If the constrained minimization problems* (18)–(41) *has an optimal solution, the solution* θ_n^* *is an unbiased estimator of the population value.*

Proof. The optimal solution of the constrained minimization problems (18)–(41) is the sample estimator $\theta_n^* = \tilde{\theta}_n$ and is obtained as a rational function of the sample moments. By Corollary 3.1 and Lemma 3.1 the sample values converge in probability to the population values. □

The constraints (37)–(40) are sample moments of the probability distribution function of the residuals which are made to assume given values in terms of the variance σ^2 and its higher powers. These constraints enforce the residuals to have a noninformative distribution, here a Gaussian, fact reinforced by the next result:

THEOREM 3.2. [50]. *Let the constrained minimization problems* (18)–(41) *have a solution and let the regression function and its derivatives up to the third order with regard to all arguments be bounded; then* $\left(\tilde{\theta} - \theta\right)\sqrt{n}$ *is normally distributed as the sample size* $n \to \infty$.

Thus the condition (6) is also met. In particular, notice that if the conditions of Theorem 3.2 are not met, the sample values of the estimated parameters from the true parameter may not be distributed according to the normal distribution requiring different tests of significance and test of hypotheses. However, Conditions (1) and (2) will be satisfied.

Condition (3), which is also very important will hold in all cases that the constrained minimization problems (18)–(41) has a solution, as the next theorem shows.

THEOREM 3.3. [50]. *Let the constrained minimization problems* (18)–(41) *have a solution then the estimator* $\tilde{\theta}$ *is asymptotically efficient.*

Finally a result can be presented showing that this constrained minimization problem (18)–(41) will dominate the solutions obtainable by the traditional three phase procedure, since whenever the latter has a solution, the new procedure will also have a solution, but not converely.

THEOREM 3.4. *Let the given optimal control problem as described in* (13)–(15) *have a unique solution and let all the conditions of Jennrich* [34] *be met, so that the solution of the maximum likelihood estimate of the unconstrained problem is well defined. Then, the solution of optimal control problems* (13)–(15) *will be equal to the solution of the constrained optimization problems* (18)–(41).

Proof. For a given set of functional forms the two sets of estimates will be equal, as the conditions of Jennrich are equivalent to the sets of constraints (19)–(41). Thus the resultant optimization problems are identical and their solution, as it is unique, will be the same. □

4. Convergence Results

The aim of this section is to describe an iterative procedure to minimize a given function subject to equality and inequality constraints, by solving a linear complementarity problem at each iteration, subject to a suitable trust region defined by a set of inequalities and to prove its general convergence conditions. These proofs are directly relevant to the optimal control problem that has been formulated, since it provides the conditions under which it can be solved.

Many successive (recursive) quadratic programming methods have been proposed to solve the general constrained nonlinear optimization problem, all of which use active set strategies and suitable line search algorithms.

The existence of a general solution procedure for the linear complementarity problem [60] permits incorporating this algorithm recursively in an optimization algorithm and so to avoid the use of active set strategies, to handle inequality constraints. This may be beneficial if the quadratic objective function is not convex.

The determination of a step-length by an appropriate line search algorithm also often proves troublesome for general problems. The implementation of trust region algorithms for unconstrained optimization can partly offset some of these difficulties and, moreover, global convergence results may be given under very mild assumptions on the function.

By using the linear complementarity algorithm, mentioned above, in a recursive quadratic programming framework with a trust region formed as a box around the iteration point, when convenient, all the best features of the various methods can be obtained and therefore powerful results and a fast easily implementable algorithm can be defined.

Consider the following optimization problem:

$$\text{Min} \quad Z = f(w) \quad f: R^n \to R, \tag{42}$$

$$g(w) \geqslant 0 \quad g: R^n \to R^p, \tag{43}$$
$$h(w) = 0 \quad h: R^n \to R^q. \tag{44}$$

The proposed algorithm consists in defining a quadratic approximation to the objective function, a linear approximation to the constraints and determining a critical point of the approximation by solving a linear complementarity problem, as given in [60].

Expanding the functions in a Taylor series, at the given iteration point w^k, the equality constraints may be eliminated simply by converting them into $p+1$ inequality constraints. Thus:

$$h(w) = h(w^k) + \nabla h(w^k)(w - w^k) \geqslant 0, \tag{45}$$
$$-e_q^T h(w) = -e_q^T (h(w^k) + \nabla h(w^k)(w - w^k)) \geqslant 0. \tag{46}$$

Unconstrained variables must be transformed into nonnegative variables for the LCP algorithm. So let:

$$\zeta = \mathrm{Inf}\{w_i \mid w_i \in \{g(w) \geqslant 0, h(w) = 0\}\}, \tag{47}$$

where ζ is a suitable lower bound to the unrestricted variables, which will be expressed as:

$$x_i = w_i - \zeta \geqslant 0. \tag{48}$$

Should there be no lower-bound specifiable for a variable, then as it is well known, the variable can be represented as the difference of two nonnegative variables.

A set of trust region constraints can be imposed on the problem as a system of linear inequalities centered around the iteration point, to limit the change in the possible solution. Note that such a set of inequalities has quite different properties to the usual trust region constraint imposed in Nonlinear Optimization [15]:

$$Dx + d \geqslant 0, \tag{49}$$

where $D \in R^{n \times n}$ is a suitable matrix which may be changed at every iteration and $d \in R^n$ a suitable vector. These can be included in the inequalities, so the problem to be solved iteratively is:

$$\begin{aligned}\text{Min } f(x) = {} & f(x^k + e_n\zeta) + \nabla f(x^k + e_n\zeta)(x - x^k) \\ & + \tfrac{1}{2}(x - x^k)^T \nabla^2 f(x^k + e_n\zeta)(x - x^k), \end{aligned} \tag{50}$$

$$\text{s.t. } g(x) = g(x^k + e_n\zeta) + \nabla g(x^k + e_n\zeta)(x - x^k) \geqslant 0, \tag{51}$$

$$x \geqslant 0, \tag{52}$$

where $g\colon R^n \to R^{n+p+q+1}$ and the constant terms due to the transformation can be disregarded in the derivations.

To find an initial feasible solution consider the following optimization problem:

$$\text{Min } v^T v, \tag{53}$$

$$\text{s.t. } g(x) + v \geqslant 0, \tag{54}$$

$$x \geqslant 0, \tag{55}$$

$$v \geqslant 0, \tag{56}$$

where $g\colon R^n \to R^{p+q+1}$.

It is immediate that an optimal solution to the above problem, if it results in a value of the objective function equal to zero is a feasible solution to the main problems (42)–(44). The optimization problems (53)–(56) may be solved by the same procedure as the original problem, which will be described below. The algorithm converges to a global minimum, as it will be shown, so that if the objective function value of (53) is different from zero, the original problems (42)–(44) will have no feasible solution.

Moreover, a feasible solution to the problem (42)–(44), if it exists, may be obtained with the algorithm for any starting point x' by determining suitable values to the elements of the vector $v \geqslant 0$ such that each constraint is feasible.

The two resulting quadratic problems (50)–(52) and (53)–(56) when transformed into linear complementarity problems can be solved by a linear programming routines, as indicated in [60]. The new solution point to problems (50)–(52) will always exist, whenever the trust region is included in the problem and will lie either inside the trust region or on a trust region constraint.

Whenever this point occurs inside the trust region, then it is an approximate stationary point. If the solution point occurs on a trust region constraint and the solution is feasible while a reduction in the objective function has occurred, the solution point is taken as the new starting point and a new iteration is started. Otherwise, if the new point is infeasible, the trust region is reduced. Finally if there has been an increase in the objective function, the trust region is enlarged and the iteration is repeated, with suitable safeguards to provoke a reduction in the objective function.

If the objective function is bounded [3], for all values of the variables which satisfy the constraints, then a local minimum point will be found eventually.

Consider the optimization problem given in (42)–(44) and assume that the objective function $f \in C^2$ while $g, h \in C^1$. Notice that the domain of the optimization problem is over R^n a convex space.

The problem can be written, without loss of generality as given in (50)–(52) and the Kuhn–Tucker points for this problem will be given by determining suitable solutions to the following system:

$$\nabla L(x, \lambda) = \nabla f(x) - \nabla g(x)^T \lambda \geqslant 0, \tag{57}$$

$$g(x) \geqslant 0, \tag{58}$$

$$x \geqslant 0, \tag{59}$$

$$\lambda \geqslant 0, \tag{60}$$

$$\left(x^T, \lambda^T\right) \begin{pmatrix} \nabla f(x) - \nabla g(x)^T \lambda \\ g(x) \end{pmatrix} = 0, \tag{61}$$

which for simplicity, it is desired to represent in the following way.
Let

$$F(z) = \begin{pmatrix} \nabla f(x) - \nabla g(x)^T \lambda \\ g(x) \end{pmatrix}, \tag{62}$$

$$z^T = \left(x^T, \lambda^T\right), \tag{63}$$

then:

$$F(z) \geqslant 0, \tag{64}$$

$$z \geqslant 0, \tag{65}$$

$$z^T F(z) = 0. \tag{66}$$

This problem can be written as a variational inequality:

$$F(z)^T (y - z) \geqslant 0. \tag{67}$$

The solutions of the two problems are identical.

THEOREM 4.1. [40]. *$z \in R^{p+q+n+1}$ is a solution to the nonlinear complementarity systems* (64)–(66) *if and only if*

$$z \geqslant 0, \quad F(z)(y - z) \geqslant 0, \quad \forall\ y \geqslant 0. \tag{68}$$

Proof. ($\Rightarrow$) Let z be a solution to the nonlinear complementarity problem. Then by (67) $F(z)^T y \geqslant 0, \quad \forall y \geqslant 0$. Thus

$$F(z)^T (y - z) = F(z)^T y - F(z)^T z \geqslant 0. \tag{69}$$

($\Leftarrow$) Let $z \geqslant 0$ be a solution to (67) and consider the vector $y = z + e_i$ where e_i is the unit vector in the chosen space. Thus $F(z)_i \geqslant 0 \quad \forall i$ and thus $F(z) \geqslant 0$.

Consider now $y = 0$, there follows

$$F(z)^T (y - z) = F(z)^T z, \leqslant 0, \tag{70}$$

but as $F(z) \geqslant 0$ and $z \geqslant 0$, it follows that $F(z)^T z = 0$. □

There exists an equivalence also between a solution to a variational inequality and a fixed point of a mapping.

THEOREM 4.2 [54]. *Let $K \subseteq R^n$ be a closed convex set. Then, for every $x \in R^n$ there exists a unique point y such that: $\|x - y\| \leqslant \|x - z\|, \quad \forall z \in K$. The point y is the orthogonal projection of x on K with respect to the Eucledian norm, i.e. $y = Pr_K x = \operatorname{argmin}_{z \in K} \|x - z\|$.*

THEOREM 4.3 [54]. *Let $K \subseteq R^n$ be a closed convex set, then $y = Pr_K x$ if and only if $y^T (z - y) \geqslant x^T (z - y), \quad \forall z \in K$.*

THEOREM 4.4 *Let $K \subseteq R^n$ be a closed convex set, then $z^* \in K$ is a solution to the variational inequality if and only if for any $\gamma > 0$, z^* is a fixed point, such that:*

$$z^* = Pr_K \left(z^* - \gamma F\left(z^*\right)\right). \tag{71}$$

Proof. ($\Rightarrow$) Let z^* be a solution to the variational inequality $F(z^*)^T (y - z^*) \geqslant 0, \forall y \in K$. Multiply this inequality by $-\gamma < 0$ and add $(z^*)^T (y - z^*)$ to both sides of the resulting inequality. There results:

$$\left(z^*\right)^T \left(y - z^*\right) \geqslant \left(z^* - \gamma F\left(z^*\right)\right)\left(y - z^*\right), \quad \forall y \in K \tag{72}$$

and therefore, $z^* = Pr_K (z^* - \gamma F(z^*))$.

($\Leftarrow$) If $z^* = Pr_K (z^* - \gamma F(z^*))$ for $\gamma > 0$, then

$$\left(z^*\right)^T \left(y - z^*\right) \geqslant \left(z^* - \gamma F\left(z^*\right)\right)\left(y - z^*\right), \quad \forall y \in K \tag{73}$$

and so $F(z^*)^T (y - z^*) \geqslant 0, \quad \forall y \in K$. □

Consider the application $F\colon R^n \to R^n$ and expand it in a Taylor series around a point $z' \in R^n$ to get:

$$F(z) = F\left(z'\right) + \nabla F\left(z'\right)\left(z - z'\right) \tag{74}$$

then for any $\varepsilon_1 > 0$ there exists a scalar $r > 0$ such that:

$$\left\| F(z) - F\left(z'\right) + \nabla F\left(z'\right)\left(z - z'\right)\right\| \leqslant \varepsilon_1 \left\|z - z'\right\|, \quad \forall \left\|z - z'\right\| \leqslant r \tag{75}$$

as it has been proved [23].

Thus in a small enough neighbourhood, the approximation of the nonlinear complementarity problem by a linear complementarity problem will result sufficiently accurate, so that instead of solving systems (64)–(66), the linear complementarity system approximation can be solved. Recall that by construction, the subspace of the Euclidean space is bounded and closed, so that the following lemma can be applied, both to the nonlinear complementarity problem and to the linear complementarity approximation of it.

LEMMA 4.1 [33]. *Let $K \subset R^n$ be a nonempty, convex and compact set and let $F\colon K \to K$ be a continuous mapping. Then there exists a fixed point $z^* \in K$ for this mapping.*

THEOREM 4.5 [27]. *Given the nonlinear complementarity problems* (64)–(66) *where $F(z)$ is continuous, there exists a connected set $S \subset R^n$ such that:*

1. *Each $z \in S$ is a solution to the nonlinear complementarity problem such that $D_i^T x = k \leqslant d_i$, one of the trust region constraints, restricted by the scalar k,*
2. *For each value $k \in R_+$, there exists a solution to the nonlinear complementarity problem $z \in S$.*

COROLLARY 4.1 [47]. *Consider the linear complementarity problem representation of* (50)–(52) *and a set $S = \{\mu(t) | t \in R_+\}$ where $\mu\colon R_+ \to R$ is a piecewise continuous mapping then:*

1. *Each $\mu \in S$ is a solution to the linear complementarity problem, restricted to the subset $D_i^T x = k$, so one of the trust region constraints is binding.*
2. *For each $k \in R_+$ there exists an $x \in S$ which is a solution to the linear complementarity problem.*

It has been shown that every linear complementarity problem can be solved, or a solution can be shown not to exist by solving an appropriate parametric linear programming problem in a scalar variable [60]. The algorithm will find the solution of the linear complementarity problem, if such a solution exists such that $\|x\| \leqslant \alpha$, for some constant $\alpha > 0$, or declare that no solution exists, so bounded. In this case the bound can be increased.

The convergence of our algorithm can now be demonstrated. Consider a point $x' \in R^n$ such that $F(x') \geqslant 0$ and therefore feasible. Determine a neighbourhood, as large as possible, which can be indicated by:

$$Q = \left\{ z \mid \left\| z - z' \right\| \leqslant r \right\}, \tag{76}$$

where r is the coefficient defined above in (75).

Suppose that the acceptable tolerance to our solution is ε_2 so that if $(z^*)^T F(z^*) \leqslant \varepsilon_2$ then the solution is accepted. In this case, impose that:

$$\varepsilon_1 r \leqslant \frac{\varepsilon_2}{\alpha}. \tag{77}$$

The local convergence of the algorithm is established in the following theorem.

THEOREM 4.6 *If the linear complementarity problem has a solution* z^* *where all the trust region constraints are not binding, then such a solution is also a solution to the nonlinear complementarity problems* (64)–(66) *for which* $F(z^*) \geqslant 0$ *and* $(z^*)^T F(z^*) \leqslant \varepsilon_2$.

Proof. Consider the solution z^* of the linear complementarity problem (57)–(61). Recall that $\alpha \geqslant e^T z^*$ by construction and without loss of generality, take $\alpha > 1$. Consider this solution applied to the nonlinear complementarity problem, there will result:

$$\left\| F\left(z^*\right) - F\left(\hat{z}\right) + \nabla F\left(\hat{z}\right)\left(z^* - \hat{z}\right) \right\| \leqslant \varepsilon_1 \left\| z^* - \hat{z} \right\| \leqslant \varepsilon_1 r < \varepsilon_2. \tag{78}$$

For the complementarity condition

$$(z^*)^T F(z^*) = (z^*)^T \left(F(z^* - F(\hat{z}) + \nabla F(\hat{z})(z^* - \hat{z}) \right) \leqslant \left\| z^* \right\| \varepsilon_1 r \leqslant \varepsilon_2, \tag{79}$$

which follows by the complementarity condition of the LCP and the Cauchy–Schwartz inequality. Further as $\alpha > e^T z^* > \|z^*\|$ because of the nonnegativity of the solution variables. Also $\varepsilon_1 r \leqslant \frac{\varepsilon_2}{\alpha}$ so:

$$(z^*)^T F(z^*) \leqslant \varepsilon_2. \tag{80}$$

THEOREM 4.7 *Let the objective function be bounded for the nonlinear optimization problems* (42)–(44) *and let the problem have a feasible solution, then there exists a global minimum solution.*

Proof. By Theorem 4.6 each solution to the LCP is an approximate solution to the nonlinear complementarity problems (64)–(66). By Theorem 4.5 a connected set exists such that the nonlinear complementarity problem has a solution within the trust region or on a constraint.

From this sequence choose a subsequence such that the value of the objective function, as given by Equation (42) decreases.

Since the objective function is bounded not all the solutions can lie on some trust region constraint, so a solution of the nonlinear complementarity problem which lies within the trust region constraints must eventually be determined. Let this solution be a local minimum to the nonlinear optimization problem. By repeating this procedure the global minimum will be determined. □

5. Conclusions

A SCM system should be modelled by nonlinear stochastic and dynamical systems, since otherwise eventual simplifications in the representation will lead to suboptimization and to the possibility that profitable policies are not solutions to the model, since they cannot be represented by the assumed model.

Thus the analysis must be carried out at the highest level of generality with as few a priori assumptions as possible. Consequently managerial insights and anectodotical evidence should be avoided, so a data driven modelling system should be used.

As the representation will be dynamic and nonlinear, great complexities in the modelling of such a phenomenon will be encountered and it is shown in this paper how to overcome these complexities and obtain a syntactically correct solution, which is also semantically adequate.

Since stochastic disturbances and exogenous events will alter the trajectories of development and therefore, the realizability of the goals chosen, it is essential that the reachability of the goals be ascertained periodically. Also the controllability of the system must be checked, since after some disturbance it cannot be assured that the system is still controllable in the sense desired. Finally, if the system is unobservable then there is no indication of the policy that is being followed and if it is an optimal one.

If disturbances provoke instabilities in the system, it is important to be aware of them, as policies may have to be modified, since it could be very expensive for the firm to maintain an unstable equilibrium, such as respecting a collaborative arrangement at a set price, when all the market forces are acting towards a reduction in the price, say.

Thus dynamical systems theory seems to be indispensable as the modelling framework of a SCM.

As relations must be estimated and optimal policy variables must be determined, to avoid potential suboptimization and biased estimates, a combined estimation and optimization algorithm must be applied to solve simultaneously these two problems. In fact, not only can these two problems be posed in a simultaneous fashion, but convergence results can be given, which guarantee that the estimates will have the required properties and that the optimal policy is the best one that can be formulated on the basis of the evidence. The certainty equivalence results guarantee that the policy chosen for the first period coincides with the one obtainable on the basis of a deterministic problem rather than one subject to uncertainties.

In this work, we have examined the properties of such an approach in an abstract setting, but its implementation to actual cases, should be apparent and has been presented elsewhere, [21]. Starting from analytical accounting and budgeting systems and preliminary specifications of the functional relations among the control variables, initial models are obtained, which through iterative refinement, better and better models can be specified. Through the dynamical models with the state space which is endogenously determined, unobservable variables will be introduced in the system, which can later be interpreted.

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